Kicked-rotor quantum resonances in position space

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We present an approach of the kicked-rotor quantum resonances in position space, based on its analogy with the optical Talbot effect. This leads to a very simple picture of the physical mechanism underlying the dynamics and to analytical expressions for relevant physical quantities. The ballistic behavior, which is closely associated with quantum resonances, is analyzed and shown to emerge from a coherent adding of successive kicks thanks to a periodic reconstruction of the spatial wave packet.

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I. INTRODUCTION

The kicked rotor has played a central role in studies of classical and "quantum chaos" (which is defined as the quantum behavior of a system whose classical counterpart is chaotic). A kicked rotor (KR) is formed by a particle orbiting a fixed circular orbit to which an instantaneous force (a kick) is applied periodically. In 1995, Raizen and co-workers [1] established this system as a privileged ground for studies of quantum chaos by realizing experimentally a kicked rotor in the quantum regime with laser-cooled atoms placed in a kicked (i.e., rapidly turned on and off) laser standing wave.

Despite its apparent simplicity, the quantum kicked rotor (QKR) has very remarkable dynamical properties. One of the most studied is the so-called "dynamical localization:" in contrast to the classical case, the ergodic diffusion does not last forever in the quantum case. After a characteristic "localization time," the diffusion is stopped by destructive quantum interferences [1–4]. Another feature of the QKR that has been the object of a recent burst of activity is the existence of quantum resonances (QRs), whose most dramatic manifestation is the appearance of a ballistic motion.

The experimental realization of the QKR has triggered in the last decade an impressive number of studies involving dynamical localization [5–11], quantum transport [12–15], ratchets [16–19], chaos-assisted tunneling [20,21], and classical and quantum resonances [22–27]. Quantum resonances have been used in studies of fundamental aspects of quantum chaos such as quantum stabilization [28] or measurements of the gravitation [29]. "High-order" quantum resonances were also observed recently [30–32] both with laser-cooled atoms and a Bose-Einstein condensate.

In the present work, we show that quantum resonances have a very simple and intuitive interpretation in position space, as opposed to the more common momentum space representation (see Ref. [33], and references therein). QRs in position space have been previously considered by Izrailev and Shepelyansky [3,34]. Here, we extend their approach and build a simple physical picture that enlightens the underlying physics and allows the calculation of quantities of experimental relevance, such as average momentum or the kinetic energy.

II. PHYSICAL ORIGIN OF QUANTUM RESONANCES

The atom-optics realization of a (quantum) kicked rotor consists of laser-cooled atoms placed in a far-detuned standing wave. The standing wave is pulsed periodically, being on for a time interval so short that the motion of the atoms can be neglected. In such conditions, the pulse can be considered as a δ function, and the atoms feel the light intensity as a kick related to a mechanical potential affecting their center-of mass motion [35–37]. The standing wave sinusoidal modulation of intensity generates a spatial potential in $\sin(2k_L x)$, where $k_L=2\pi/\lambda_L$ is the wave number of the radiation. We shall use a normalized coordinate $X=2k_L x$ which plays in fact the role of a cyclic variable: the spatial periodicity of the potential implies that the physics is invariant under translations by a multiple of $\lambda_L/2$. We can thus use either a "linear" ("unfolded") representation of the KR, or a "cyclic" or "folded" representation using the angular variable $\theta=X[\mathrm{mod}\ 2\pi]$ [38].

Choosing units such that the mass of the particle is $M=1$ and the time period of the forcing is $T=1$, the Hamiltonian of the KR is

$$H=\frac{P^2}{2}+K\cos X\sum_{n=0}^{N-1}\delta(t-n), \tag{1}$$

where P is the momentum (scaled by a factor $M/2k_L T$), X the position in the periodic potential, K (usually called "stochasticity parameter") the intensity of the kicks, and n a discrete time corresponding to the nth kick (n is an integer in normalized units). Labeling X_t, P_t the position and momentum immediately after the tth kick, and integrating the classical Hamilton equations of motion corresponding to Eq. (1) produces the so-called Chirikov's "standard map" [39]:

$$X_{t+1}=X_t+P_t, \quad P_{t+1}=P_t+K\sin X_{t+1}. \tag{2}$$

For small K the dynamics is mostly regular, except for limited zones of the phase space. As K approaches a critical value $K_c\approx 0.9716$, small separated chaotic regions appear. At the critical value K_c, chaotic regions connect to each other, leading to a diffusive behavior in momentum space. The chaotic region grows with K and for $K\gtrsim 5$ the classical dynamics tends to be ergodic in the phase space (for a more complete discussion of dynamical issues, see Ref. [40]). The average kinetic energy then increases linearly with time (or kick number t): $\langle P^2\rangle=D_c t$, where D_c is a diffusion coefficient that can be explicitly calculated [41]; to a first approximation $D_c\approx K^2/2$.

If we set $K=2\pi$ and consider a particle with initial conditions $X_0=\pi/2$ and $P_0=0$, iteration of Eqs. (2) shows that $P_t=2\pi t$: The momentum increases linearly with time; hence the kinetic energy $P^2/2$ increases quadratically, which is a signature of a ballistic dynamics, called a (classical) resonance. The origin of the ballisticity is easily seen: for this particular value of K and of the initial conditions, the particle is always kicked at the same position (modulus 2π), that is $\sin X_t=\sin[2\pi(t^2-t)+\pi/2]=1$, so that it receives a constant amount K of linear momentum per kick. The classical resonance has been experimentally observed in the atom-optics realization of the KR [22]. There are also classical antiresonances: For the above initial conditions and $K=3\pi/2$, successive kicks have opposite directions, and the momentum jumps endlessly between two values $P_t=0$ and $P_{t+1}=3\pi/2$.

In order to study the quantum case, let us consider the one-period evolution operator, or Floquet operator

$$U=e^{-iH/\hbar\!\!\!{}^{-}}=\exp\left(-i\frac{K}{k\!\!\!{}^{-}}\cos X\right)\exp\left(-i\frac{P^2}{2k\!\!\!{}^{-}}\right), \tag{3}$$

where $k\!\!\!{}^{-}=4\hbar k_L^2T/M$ is the normalized Planck's constant resulting from the definition of normalized variables satisfying the commutation relation $[X,P]=ik\!\!\!{}^{-}$. Because of the δ-function time dependence, this operator factorizes in the product of a kick operator by a free evolution operator.

The kick operator is periodic in space, so its eigenstates have a Bloch wave (BW) structure. If one starts with an initial state of well-defined momentum $|P_0\rangle$, only states of the form $|P_0+mk\!\!\!{}^{-}\rangle$ (m integer) will appear in the dynamics. We can write any arbitrary momentum P in the form $P=(m+\beta)k\!\!\!{}^{-}$ with $\beta\in[-1/2,1/2)$. This introduces the quasimomentum $k\!\!\!{}^{-}\beta$ which is thus a constant of motion. The particle wave function $\psi(X)=\langle X|\psi\rangle$ can be written as

$$\psi(X)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\tilde{\psi}(k)dk, \tag{4}$$

where $k=P/k\!\!\!{}^{-}$, that is,

$$\psi(X)=\frac{1}{\sqrt{2\pi}}\int_{-1/2}^{+1/2}d\beta\sum_{m=-\infty}^{+\infty}\tilde{\psi}_\beta(m)e^{i(m+\beta)X}=\int_{-1/2}^{+1/2}d\beta\psi_\beta(X). \tag{5}$$

The quasimomentum component $\psi_\beta(X)$ is defined by

$$\psi_\beta(X)=\frac{e^{i\beta X}}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}\tilde{\psi}_\beta(n)e^{imX}, \tag{6}$$

where the function $\sum_{m=-\infty}^{\infty}\tilde{\psi}_\beta(m)e^{imX}$ is 2π periodic.

Let us take, for simplicity, an initial state of quasimomentum $\beta=0$ and apply to it the free evolution $\exp(-ik\!\!\!{}^{-}P^2/2)$; one obtains at time $t=1^-$ (that is, just before the first kick):

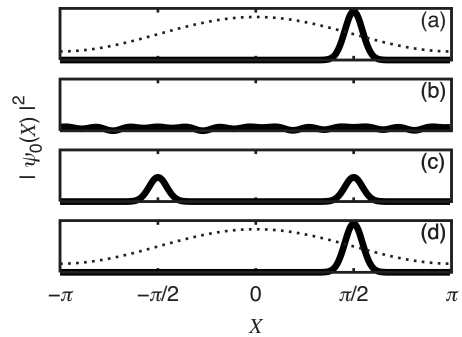

FIG. 1. Free evolution of a localized wave packet ($\beta=0$) between two kicks for $k\!\!\!{}^{-}=4\pi$ (note that the packet is shown in the interval $X\in[-\pi,\pi]$, but it reproduces itself periodically outside this range). (a) An initial wave packet centered at $X=\pi/2$, (b) wave packet at $t=0.115$, showing complete delocalization, (c) wave packet at $t=0.25$, showing two replicas of the initial shape, (d) "reconstructed" wave packet at $t=1^-$, identical to the initial one. The vertical scale is the same for all the panels.

$$\psi_0(X,t=1^-)=\frac{1}{\sqrt{2\pi}}\sum_m\tilde{\psi}_0(m,t=0)\exp\left(-ik\!\!\!{}^{-}\frac{m^2}{2}\right)\exp(imX). \tag{7}$$

Setting in the above expression $k\!\!\!{}^{-}=4\pi$, for example, the argument of the exponential phase factor in Eq. (7) is seen to be an integer multiple of 2π. Hence, the free evolution over a period leaves the wavepacket invariant. Figure 1 shows a $\beta=0$ wave function which is initially spatially localized in one potential well [42] and which evolves freely with time. The wave function becomes completely delocalized but it "focalizes" back to its initial shape just before the next kick is applied. This is analogous to the optical Talbot effect: a monochromatic beam diffracted by a grating also reforms after a certain propagation length [43]. Quantum resonance is thus the atom-optics analog of the optical Talbot effect [31,44].

As the kick operator in Eq. (3) is diagonal in position representation, it simply adds a position-dependent phase to the wave function

$$\psi_0(X,t=1)=\exp(-i\kappa\cos X)\psi_0(X,t=0), \tag{8}$$

where $\kappa\equiv K/k\!\!\!{}^{-}$. As the wave packet has always the same shape when the kick is applied, it acquires the same phase from each kick, and the effect of the kicks adds coherently. Quantum resonance can thus be simply view as a constructive interference effect.

The above discussion evidences an analogy of the physical pictures underlying the quantum and the classical resonances: In both cases, for particular values of the parameters, the dynamics is such that the particle (the wave packet in the quantum case) is always kicked at the same position so that the kick effect adds to produce a linear increase of the momentum. However, the quantum and the classical resonances have a very different nature, and this analogy does not imply that the detailed dynamics in the same. In the classical case the effect is related to the intensity of the kicks K whereas in the quantum case it depends on the value of $k\!\!\!{}^{-}$ [45].

III. ANALYSIS OF "SIMPLE" QUANTUM RESONANCES

Quantum resonances obeying the condition $\hbar\!\!\!/ = 2\pi l$ (with l a positive integer) are named "simple" quantum resonances (SQRs). In this case, the shape of BW remains invariant under the free propagation over a period, the kicking period is then a multiple of the half-Talbot time (defined in classical optics), which is the condition for the optical (integer) Talbot effect [43,46,47].

A. Time evolution

Let us consider, at some (integer) time $(t-1)$ a state $\psi_\beta(X,t-1)$ corresponding to the general form of Eq. (6), and apply to it the free-evolution operator with $k\!\!\!/ = 2\pi l$:

$$\psi_\beta(X,t^-) = \exp\left(-i\frac{P^2}{2k\!\!\!/}\right)\psi_\beta(X,t-1)$$
$$= \frac{1}{\sqrt{2\pi}}\exp(-i\pi l\beta^2)\sum_m \tilde{\psi}_\beta(m,t-1)$$
$$\times\exp(-i\pi lm^2)\exp(-i2\pi lm\beta)e^{i(m+\beta)X}.$$

As lm^2 and lm have the same parity, the first exponential factor under the sum is equal to $\exp(-i\pi lm)$; it can thus be combined with the second term, yielding $\exp[-imk\!\!\!/(\beta+1/2)]$:

$$\psi_\beta(X,t^-) = \frac{1}{\sqrt{2\pi}}\exp\left(-i\frac{k\!\!\!/\beta^2}{2}\right)\sum_m \tilde{\psi}_\beta(m,t-1)$$
$$\times\exp(-imk\!\!\!/\beta')e^{i(m+\beta)X}$$

with $\beta' \equiv \beta + 1/2$. Or, using Eq. (6)

$$\psi_\beta(X,t^-) = \exp\left(i\frac{k\!\!\!/\beta(\beta+1)}{2}\right)\psi_\beta(X-k\!\!\!/\beta',t-1). \tag{9}$$

Applying now the kick operator produces a recurrence relation linking the wave packets at times t and $(t-1)$:

$$\psi_\beta(X,t) = e^{-i\kappa\cos X}\exp\left(i\frac{k\!\!\!/\beta(\beta+1)}{2}\right)\psi_\beta(X-k\!\!\!/\beta',t-1). \tag{10}$$

Thus, $|\psi(X,t)|^2 = |\psi(X-v,t-1)|^2 = |\psi(X-vt,0)|^2$. The square modulus of the wave function remains invariant immediately before each kick, except for a drift with a rate per kick [48]

$$v = k\!\!\!/\left(\beta + \frac{1}{2}\right). \tag{11}$$

Iterating Eq. (10) down to $t=0$ gives

$$\psi_\beta(X,t) = \exp[-i\kappa\Phi(X,t)]\psi_\beta(X-vt,0) \tag{12}$$

with an accumulated phase [49]

$$\Phi(X,t) = \sum_{s=0}^{t-1}\cos(X-sv). \tag{13}$$

A remarkable property is obtained if

$$v = 2\pi\frac{p}{q} \tag{14}$$

with p,q integers, i.e., $\beta = p/(q\ell) - 1/2$. The phase defined in Eq. (13) takes the values $\Phi(X,t=q)=0$ if $q\neq 1$ and $\Phi(X,t=q)=\cos X$ if $q=1$. Consider first the case $q\neq 1$. After a recurrence time $t_r = q$ the wave function is:

$$\psi_\beta(X,t_r) = \psi_\beta(X-2\pi p,0) = e^{-i2\pi p\beta}\psi_\beta(X,0), \tag{15}$$

i.e., the particle comes back to its initial state after t_r kicks, leading to a periodic evolution. The physical origin of this periodicity can be easily seen, e.g., if $v=\pi$ ($p=1$, $q=2$, $t_r=2$): From Eqs. (12) and (13), the BW evolution over two successive kicks is

$$\psi_\beta(X,2) = e^{-i\kappa\cos X}\psi_\beta(X-\pi,1) = \psi_\beta(X-2\pi,0). \tag{16}$$

Hence, the phase added by the kicks simply cancels after two kicks. This behavior is called "antiresonance," leading to a periodic evolution.

If $q=1$, $v=2\pi p$, the evolution is given by

$$\psi_\beta(X,t_r=1) = e^{-i\kappa\cos X}\psi_\beta(X-2\pi p,0)$$
$$= e^{-i2\pi p\beta}e^{-i\kappa\cos X}\psi_\beta(X,0).$$

The wave packet thus recovers its initial shape at each kick and the wave function interacts with the potential in the same position, leading to a linear increase of the average momentum, or ballistic behavior. This is the quantum resonance. Note again analogy of this physical picture with that presented in Sec. II for the classical resonance.

B. Quantum averages

Let us now focus on the time evolution of average values. From Eq. (10) we can easily obtain a recurrence relation for the position average over the BW

$$\langle X\rangle(t) = \langle\psi_\beta(t)|X|\psi_\beta(t)\rangle = \langle X\rangle(t-1) + v. \tag{17}$$

The recurrence relation for the average momentum is

$$\langle P\rangle_\beta(t) = \langle P\rangle_\beta(t-1) + K\int_{-\pi}^{\pi}dX\,\sin X|\psi_\beta(X-v,t-1)|^2$$
$$= \langle P\rangle_\beta(t-1) + K\int_{-\pi}^{\pi}dX\,\sin(X+vt)|\psi_\beta(X,t=0)|^2. \tag{18}$$

This expression can be iterated down to $t=0$, leading to

$$\langle P\rangle_\beta(t) = \langle P\rangle_\beta(0) + K\left(\sum_{n=1}^{t}\int_{-\pi}^{\pi}dX\,\sin(X+nv)|\psi_\beta(X,0)|^2\right)$$
$$= \langle P\rangle_\beta(0) + K\frac{\sin(tv/2)}{\sin(v/2)}$$
$$\times\mathrm{Im}\left(e^{i(t+1)v/2}\int_{-\pi}^{\pi}dXe^{iX}|\psi_\beta(X,0)|^2\right). \tag{19}$$

If $v=2\pi$, the modulus of the momentum increases linearly with the (stroboscopic) time t, i.e.,

$$\langle P\rangle_\beta(t) = \langle P\rangle_\beta(0) + D_1 t \tag{20}$$

with the slope

$$D_1 = K\int_{-\pi}^{\pi} dx \, \sin x |\psi_\beta(x,0)|^2 \tag{21}$$

which appears as the force averaged over the initial spatial distribution.

For $\upsilon=\pi[\mathrm{mod}\, 2\pi]$, however, destructive interference occurs. This can be seen from Eq. (19): $\langle P\rangle_\beta(t)=\langle P\rangle_\beta(0)$ (t even) or $\langle P\rangle_\beta(t)=\langle P\rangle_\beta(0)-D_1$ (t odd). In the general case ($\upsilon\neq 2\pi[\mathrm{mod}\, 2\pi]$), the momentum is frozen.

Quantum resonance effects can also be analyzed through the temporal evolution of the average kinetic energy. Using Eq. (12), one obtains

$$\langle E\rangle_\beta(t) = \langle E\rangle_\beta(t=0) + \frac{K^2}{2}\int_{-\pi}^{\pi} dX\left(\sum_{n=1}^{t}\sin(X+n\upsilon)\right)^2 \times|\psi_\beta(X,t=0)|^2 + K\int_{-\pi}^{\pi} dX\left(\sum_{n=1}^{t}\sin(X+n\upsilon)\right) \times J(X,t=0), \tag{22}$$

where we introduced the current

$$J(X,t) = i\frac{\hbar}{2}[\psi_\beta(X,t)\partial_X\psi_\beta^*(X,t) - \text{c.c.}]. \tag{23}$$

The quantum resonance case $\upsilon=2\pi[\mathrm{mod}\, 2\pi]$ then leads to an average kinetic energy increasing quadratically with time:

$$\langle E\rangle_\beta(t) = \langle E\rangle_\beta(t=0) + \frac{1}{2}K^2t^2\int_{-\pi}^{\pi} dX \, \sin^2 X|\psi_\beta(X,t=0)|^2 + Kt\int_{-\pi}^{\pi} dX \, \sin X J(Xx,t=0). \tag{24}$$

The ballistic growth is seen to be proportional to the quantum average of $K^2 \sin^2 X$.

C. Map

The dynamics of position and momentum averages at "stroboscopic" times t can be described by a map. We assume that the initial BW is sharply localized around its mean position $\langle X\rangle_\beta(t=0)=X_0$. Noting $P_t=\langle P\rangle_\beta(t)$ and $X_t=\langle X\rangle_\beta(t)$ we get from Eq. (18) [taking $\int_{-\pi}^{\pi}dX|\psi_\beta(X,t=0)|^2=1$]:

$$P_t = P_{t-1} + K\sin X_t, \tag{25a}$$

$$X_t = X_{t-1} + \upsilon. \tag{25b}$$

Unlike the standard map [Eq. (2)], Eq. (25b) does not depend on P_t but only on υ. Thus, after t kicks

$$X_t = X_0 + \upsilon t, \quad P_t = P_{t-1} + K\sin(X_0+\upsilon t). \tag{26}$$

Iterating this equation produces

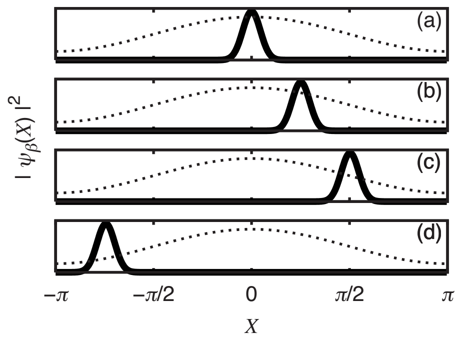

FIG. 2. Wave packet plotted at different integer times, for $\hbar=4\pi$ and an irrational quasimomentum ($\beta=\frac{\pi}{50}\approx 0.063$). (a) Initial wave packet ($|\psi_\beta|^2$) centered at $X=0$, (b) $t=1$, (c) $t=2$, (d) $t=5$. The shape of the potential is represented in dotted lines. The wave packet interacts with the potential at different positions, and the average effect tends to zero.

$$P_t = P_0 + K\sum_{n=1}^{t}\sin(X_0+n\upsilon). \tag{27}$$

Ballisticity is found if $\upsilon=2\pi(\mathrm{mod}\, 2\pi)$:

$$P_t = P_0 + tK\sin X_0 = P_0 + D_1 t \tag{28}$$

with

$$D_1 = K\sin X_0. \tag{29}$$

The quantum resonance condition thus depends on the quantum parameters $\hbar$ and β (through υ), but the kick intensity K and the initial position X_0 determine only the growth rate of the average momentum.

Other kinds of behavior can be found. Generally, rational quasimomentum values $\beta=p/q$ produces periodic motion (antiresonance) with period $t_r=q$ due to kick compensation effect discussed above. In Sec. III A we showed, e.g., that if $\hbar=2\pi$ and $\beta=0$ ($\upsilon=\pi$), the motion is periodic with a recursion time $t_r=2$. For irrational quasimomenta, the wave packet is kicked at different positions and does not display resonant behaviors (Fig. 2).

IV. THE $\hbar=\pi$ HIGH-ORDER RESONANCE

"High-order" quantum resonances are the quantum-mechanical analogs of the fractional optical Talbot effect [50]. In this case, after a free propagation the initial packet does not reconstruct in an identical packet, but forms two or more replicas of the original one. The action of the kick on these subpackets generates different quantum phases producing quantum interference effects. This makes high-order quantum resonances fundamentally different from, and more complex than, simple ones.

High-order quantum resonances (HQRs) correspond to a dimensionless Planck's constant of the form $\hbar=4\pi r/s$, with r and $s>2$ integers. A reasoning similar to that leading to Eq. (10) shows that after the free propagation the initial packet refocalizes to s uniformly spaced subpackets if s is odd, and $s/2$ subpackets if s is even. We shall consider here only the simplest case $\hbar=\pi$. The method introduced below can in principle be generalized to more complicated cases, but the resulting algebra is cumbersome.

Using Eq. (6) and applying the free-evolution operator with $\hbar=\pi$ produces

$$\psi_\beta(X,t^-)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{i\pi\beta^2}{2}\right)\sum_n \tilde{\psi}_\beta(n,t-1)\times\exp\left(-\frac{i\pi n^2}{2}\right)\exp(-i\pi n\beta)\exp[i(n+\beta)X]. \tag{30}$$

We show in Appendix A that the above expression can be written as

$$\psi_\beta(X+wt,t)=\frac{e^{-i\kappa\phi(X,t)}}{\sqrt{2}}[e^{-i\pi/4}\psi_\beta(X+w(t-1),t-1)+e^{i\pi/4}e^{i\beta\pi}\psi_\beta(X+w(t-1)-\pi,t-1)], \tag{31}$$

where w is the packet (stroboscopic) drift rate defined by $w=\hbar\beta$ and ϕ is a "local" phase

$$\phi(X,t)=\kappa\cos(X+wt). \tag{32}$$

Equation (31) shows that at any (integer) time t, $\psi_\beta(X+wt,t)$ is the superposition of two subpackets having the same shape as the initial BW, centered at $X=0$ and $X=\pi$. Each subpacket is multiplied by a phase factor which is the sum of the accumulated phases, producing a complex interference pattern. The principle of our calculation is thus to keep track of the coefficient of each subpacket, as their shape is fixed. We thus write

$$\psi_\beta(X+wt,t)=c_1(X,t)\psi_\beta(X,0)+c_2(X,t)\psi_\beta(X-\pi,0), \tag{33}$$

where $c_1(X,t)$ and $c_2(X,t)$ are 2π-periodic complex amplitudes. We are thus considering a coupled two-level system initially in "level 1" ($c_1=1$ at $t=0$) which is progressively "transferred" to "level 2" and then back again to "level 1," performing the analogous of a Rabi oscillation.

Define the state vector

$$\mathbf{c}_t=\begin{pmatrix} c_1(X,t)\\ c_2(X-\pi,t)\end{pmatrix}. \tag{34}$$

We show in Appendix A 2 that this vector obeys the recurrence relation

$$\mathbf{c}_t=e^{-i\pi/4}\mathbf{M}_t\cdot\mathbf{c}_{t-1}, \tag{35}$$

where $\mathbf{M}_t$ is a matrix depending on time and space

$$\mathbf{M}_t=\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi} & ie^{-i\phi}e^{-i\beta\pi}\\ ie^{i\phi}e^{i\beta\pi} & e^{i\phi}\end{pmatrix} \tag{36}$$

via $\phi=\phi(X,t)$, Eq. (32). The matrix $\mathbf{M}_t$ can be recast as

$$\mathbf{M}_t=\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi} & 0\\ 0 & e^{i\phi}\end{pmatrix}\begin{pmatrix} 1 & ie^{-i\beta\pi}\\ ie^{i\beta\pi} & 1\end{pmatrix}. \tag{37}$$

The rightmost matrix in the above product stands for the free propagation that induces a coupling between the two subpackets; it is thus responsible for the interference effects. The leftmost one represents the effect of the kick and is diagonal in the x representation.

Analytical results can be obtained in the case $\beta=0$, where the matrix $\mathbf{M}_t$ is time independent. It is then easy to write $\mathbf{c}_t$ as a function of the initial condition $\mathbf{c}_0=\binom{1}{0}$:

$$\mathbf{c}_t=e^{-it\pi/4}[\mathbf{M}_t]^t\mathbf{c}_0.$$

The eigenvalues of $\mathbf{M}_t$ are

$$\lambda=\exp(\pm i\Theta), \tag{38}$$

where the phase Θ is given by

$$\cos\Theta=\frac{\cos\phi}{\sqrt{2}}=\frac{\cos(\kappa\cos X)}{\sqrt{2}}$$

(note that $\pi/4\leq\Theta\leq 3\pi/4$). If $\mathbf{P}$ is the diagonalizing matrix, then (see Appendix A 3)

$$\mathbf{c}_t=e^{-it\pi/4}\mathbf{P}\begin{pmatrix} e^{it\Theta} & 0\\ 0 & e^{-it\Theta}\end{pmatrix}\mathbf{P}^{-1}\mathbf{c}_0. \tag{39}$$

After a straightforward calculation, one founds

$$c_1(X,t)=e^{-it\pi/4}\left[\cos(t\Theta)-\frac{i\sqrt{2}}{2}\frac{\sin\phi}{\sin\Theta}\sin(t\Theta)\right] \tag{40}$$

and

$$c_2(X-\pi,t)=\frac{i\sqrt{2}}{2}\frac{e^{i\phi}}{\sin\Theta}e^{-it\pi/4}\sin(t\Theta). \tag{41}$$

The average momentum is then (see Appendix B):

$$\langle P\rangle_\beta(t)=\langle P\rangle_\beta(t-1)+K\int_{-\pi}^{\pi}dX\sin X|\psi_\beta(X,t)|^2. \tag{42}$$

In the general case, it is difficult to analyze the behavior of the amplitudes c_1 and c_2 since they depend implicitly on X in a quite complicated way. Analytic results can, however, be obtained assuming that the BW $\psi_\beta(X,t=0)$ is sharply localized at some position X_0 in the interval $[-\pi,\pi]$ (with its width $\Delta X\ll 2\pi$):

$$\langle P\rangle_\beta(t)=\langle P\rangle_\beta(t-1)+K\int_{-\pi}^{\pi}dx\sin(X+wt)\times[|c_1(X,t)|^2-|c_2(X-\pi,t)|^2]|\psi_\beta(X,0)|^2. \tag{43}$$

This expression is valid if the two subpackets do not overlap significantly and shows that if the two subpackets have the same weight, the overall momentum shift per kick is zero: The subpackets are localized around positions X_0 and $X_0+\pi$, and subjected to opposite forces $+K\sin(X_0)/2$ and $-K\sin(X_0)/2$.

For a wave packet strongly localized at X_0, the amplitudes in the decomposition (43) can be evaluated at $X=X_0$ and

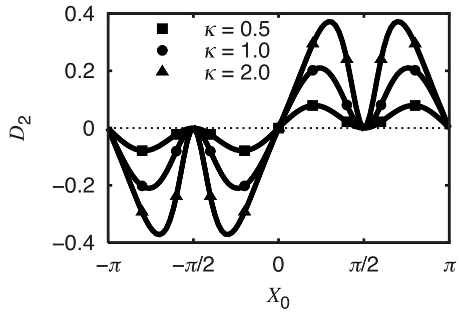

FIG. 3. Momentum slope D_2 [Eq. (48)] as a function of X_0 for three values of κ: 0.5 (squares), 1.0 (circles), and 2.0 (triangles).

depend only on time, while the phases ϕ and Θ take constant values ($\phi=\kappa\cos X_0$). Hence, the average momentum evolution is

$$\langle P\rangle_\beta(t)=\langle P\rangle_\beta(t-1)+K\sin(X_0+wt)[1-2|c_2(X_0-\pi,t)|^2], \tag{44}$$

which, in the case $\beta=0$, takes the explicit form

$$\langle P\rangle_{\beta=0}(t)=\langle P\rangle_{\beta=0}(t-1)+K\sin X_0\left(1-\frac{\sin^2(t\Theta)}{\sin^2\Theta}\right). \tag{45}$$

The expression inside parentheses is characteristic of the diffraction by a grating. Iterating down to $t=0$ produces

$$\langle P\rangle_{\beta=0}(t)=\langle P\rangle_{\beta=0}(t=0)+K\sin X_0\left[\left(\frac{\sin^2\phi}{1+\sin^2\phi}\right)t+\frac{1}{1+\sin^2\phi}\left(\frac{\sin[(2t+1)\Theta]}{2\sin\Theta}-\frac{1}{2}\right)\right]. \tag{46}$$

The average momentum evolution is seen to be, for arbitrary ϕ and Θ, a mix of two different behaviors: ballisticity, corresponding to the first term in the brackets, and oscillation, described by the second term. This also contrasts with SQRs, where the dynamics is either ballistic or oscillatory.

For long times and $\phi\neq 0$, the ballistic term dominates and one gets

$$\langle P\rangle_{\beta=0}(t)=\langle P\rangle_{\beta=0}(t=0)+D_2t \tag{47}$$

with

$$D_2=K\sin X_0\left(\frac{\sin^2\phi}{1+\sin^2\phi}\right)=D_1\left(\frac{\sin^2\phi}{1+\sin^2\phi}\right) \tag{48}$$

being the rate of change of the average momentum. The behavior of D_2 as a function of X_0 is displayed in Fig. 3 for different values of κ. It is interesting to compare Eq. (29) with its counterpart for SQRs, Eq. (48): In particular, for SQRs the maximum of D_1 occurs for at $X_0=\pi/2$ (i.e., where the force is maximum), whereas in the present case $D_2=0$ if $X_0=\pi/2$, because the kicks acting on each subpackets are in such case opposite one to the other.

The coefficient D_2 depends periodically on the kick intensity κ via $\phi=\kappa\cos X_0$, as shown in Fig. 4. In contrast to the SQR, increasing the kick intensity does not necessarily increase the slope of the momentum, because of compensating effects as described above.

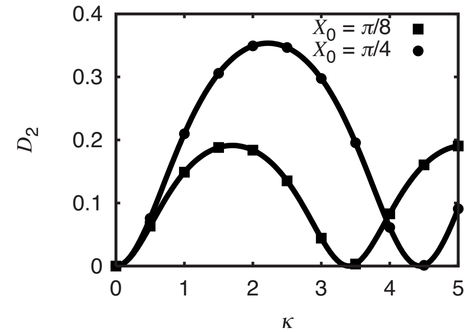

FIG. 4. Momentum slope D_2 [Eq. (48)] as a function of κ for two initial positions $X_0=\pi/4$ (circles) and $X_0=\pi/8$ (squares). Increasing κ (that is, the kick intensity) does not necessarily increase the momentum slope.

Analogous results describing the ballistic behavior for an initial state which is an eigenvector of the momentum have been obtained via a quite different approach in Ref. [51], but our localized-packet approach has the merit of providing a clearer picture of the underlying physics.

For $\beta\neq 0$, the occurrence of ballisticity depends of the periodicity on $\mathbf{M}_t$. Ballisticity will emerge if the relation $\mathbf{M}_{t+t_r}=\mathbf{M}_t$ is fulfilled for some (integer) recurrence time t_r. This happens for any rational value of quasimomentum. For example, if $\beta=1/2$, $\mathbf{M}_t$ [Eq. (36)] has a period of four kicks: $\mathbf{M}_{t+4}=\mathbf{M}_t$. One can then apply the above diagonalization method to $\mathbf{M}=\mathbf{M}_4\cdot\mathbf{M}_3\cdot\mathbf{M}_2\cdot\mathbf{M}_1$. To illustrate the resulting behavior, Fig. 5 shows the time evolution of average momentum obtained by direct integration of the Schrödinger equation. One observes essentially the same types of behavior.

V. AVERAGING OVER QUASIMOMENTUM

In experiments performed with laser-cooled atoms (not with ultracold atoms) the initial momentum distribution is larger than the Brillouin zone unless some velocity selection process is applied. A reasonable assumption is then that all

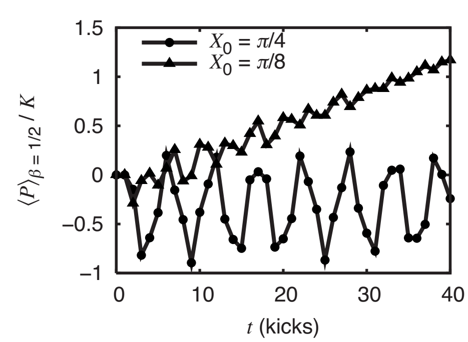

FIG. 5. Evolution of the average momentum for $\beta=1/2$ obtained by numerical integration of the Schrödinger equation. The initial wave function has the same Gaussian shape as in Fig. 1 and is centered at the positions $X_0=\pi/4$ (circles), which displays a dominant oscillatory behavior, or $X_0=\pi/8$ (triangles), displaying dominant ballisticity.

quasimomenta are equally represented in the initial state. In position space, this corresponds to an initial wave packet which is localized on a single potential well. In this simple case, for $\hbar=2\pi l$, we can work out an analytical expression for the momentum and kinetic energy averaged over the quasimomentum (see Appendix C):

$$\langle P\rangle(t)=\langle P\rangle(t=0)+K\sum_{n=1}^{t}\int_{-1/2}^{1/2}d\beta\int_{-\pi}^{\pi}dX\,\sin(X+nv)\times|\psi_\beta(X,0)|^2. \tag{49}$$

Assuming that $\psi_\beta(X,0)$ is β independent $[\psi_\beta(X,0)=\varphi(X)$ and $\int_{-\pi}^{\pi}dx|\varphi(X)|^2=1]$, and recalling that $v=2\pi l(\beta+1/2)$, the averaging over β leads to $\langle P\rangle(t)=\langle P\rangle(t=0)$. Similarly, for the kinetic energy, we get from Eq. (22)

$$\langle E\rangle(t)=\langle E\rangle(t=0)+\frac{K^2}{2}\int_{-1/2}^{1/2}d\beta\int_{-\pi}^{\pi}dX\left(\sum_{n=1}^{t}\sin(X+nv)\right)^2\times|\varphi(X)|^2+K\int_{-1/2}^{1/2}d\beta\int_{-\pi}^{\pi}dx\left(\sum_{n=1}^{t}\sin(X+nv)\right)\times J(X,t=0).$$

When we integrate over β the last term cancels out, and the only contribution comes from

$$\int_{-1/2}^{1/2}d\beta\left(\sum_{n=1}^{t}\sin(X+nv)\right)^2=\int_{-1/2}^{1/2}d\beta\left(\sum_{n=1}^{t}\sin^2(X+\pi ln+2\pi ln\beta)\right)=\frac{t}{2}. \tag{50}$$

We thus get a linear increasing of the average kinetic energy

$$\langle E\rangle(t)=\langle E\rangle(t=0)+\frac{K^2t}{4}. \tag{51}$$

Hence, the ballistic contribution, corresponding to a "null-measure ensemble" of rational quasimomenta, cannot be detected by measuring quantities averaged over quasimomentum. Experimentally, the optimum situation for observing QRs is to perform quasimomentum selection, either by using stimulated Raman transitions [32,52] or by using a Bose-Einstein condensate [14,30,31]. However, it is possible to detect QRs with atoms issued from a magneto-optical trap if one can measure the full momentum distribution with enough precision to see the ballistic parts of the wave function separating out of the diffusive part for long enough times, as experimentally evidenced by D'Arcy *et al.* [24,53].

VI. CONCLUSION

In the present work, we presented a description of quantum resonances of the kicked rotor in position space, both for simple and for high-order quantum resonances. We have shown that the dynamics can be simply understood in terms of a spatial wave packet that comes back to its in initial form after a finite number of kicks, according to the value of the quasimomentum. This picture, inspired of the atom-optics analog of the Talbot effect, provides an intuitive understanding of the underlying physics.

APPENDIX A: BLOCH WAVE EVOLUTION FOR THE $\hbar=\pi$

1. Wave function evolution

The free-propagation factor $\exp(-i\frac{\pi}{2}n^2)$ in Eq. (30) takes two different values according to the parity of n:

$$\exp\left(-i\frac{\pi}{2}n^2\right)=1(n\ \text{even})$$
$$=-i(n\ \text{odd}).$$

Inserting

$$\tilde{\psi}_\beta(n,t)=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}dXe^{-i(n+\beta)X}\psi_\beta(X,t) \tag{A1}$$

in Eq. (30) produces

$$\psi_\beta(X,t^-)=\frac{\exp(-i\pi\beta^2/2)}{2\pi}\sum_n\int_{-\pi}^{\pi}dX'e^{-i(n+\beta)(X-X')}\times\psi_\beta(X',t-1)e^{-i\pi n^2/2}e^{-i\pi n\beta}. \tag{A2}$$

One can then separate even and odd terms by setting $n=2p$ and $n=2p+1$:

$$\psi_\beta(X,t^-)=\frac{\exp(-i\pi\beta^2/2)}{2\pi}\int_{-\pi}^{\pi}dX'\psi_\beta(X',t-1)e^{-i\beta(X-X')}\times(1-ie^{-i\pi\beta}e^{-i(X-X')})\sum_p e^{-i2\pi p\beta}e^{-i2p(X-X')}. \tag{A3}$$

Using the relation $\sum_p e^{-i2\pi p\beta}e^{-i2p(X-X')}=(1/2)\sum_n\delta(X-X'-\pi\beta+n\pi)$ and integrating with respect to X', gives Eq. (31) (note that only $n=0,1$ contribute for $\beta>0$ and only $n=0,-1$ for $\beta<0$).

2. "Two-level" system

The evolution equations for the amplitudes $c_{1,2}$ are obtained in the following way. Inserting Eq. (33) calculated at time $(t-1)$ into Eq. (31) gives

$$\psi_\beta(X+wt,t)=\frac{e^{-i\phi(X,t)}}{\sqrt{2}}\{e^{-i\pi/4}[c_1(X,t-1)\psi_\beta(X,0)+c_2(X,t-1)\times\psi_\beta(X-\pi,0)]+e^{i\pi/4}e^{i\beta\pi}[c_1(X-\pi,t-1)\times\psi_\beta(X-\pi,0)+c_2(X-\pi,t-1)\psi_\beta(X-2\pi,0)]\}$$

which can be put in a simpler form [using $\psi_\beta(X-2\pi,0)=e^{-i2\pi\beta}\psi_\beta(X,0)$]:

$$\psi_\beta(X+wt,t) = \frac{e^{-i\phi(X,t)}}{\sqrt{2}}\{[e^{-i\pi/4}c_1(X,t-1) + e^{i\pi/4}e^{-i\beta\pi}c_2(X-\pi,t-1)] \times \psi_\beta(X,0) + [e^{i\pi/4}e^{i\beta\pi}c_1(X-\pi,t-1) + e^{-i\pi/4}c_2(X,t-1)]\psi_\beta(X-\pi,0)\}.$$

By comparing with Eq. (33), one finds

$$c_1(X,t) = \frac{e^{-i\phi(X,t)}}{\sqrt{2}}[e^{-i\pi/4}c_1(X,t-1) + e^{i\pi/4}e^{-i\beta\pi}c_2(X-\pi,t-1)],$$

$$c_2(X,t) = \frac{e^{-i\phi(X,t)}}{\sqrt{2}}[e^{i\pi/4}e^{i\beta\pi}c_1(X-\pi,t-1) + e^{-i\pi/4}c_2(X,t-1)].$$

The last expression can be written

$$c_2(X-\pi,t) = \frac{e^{i\phi(X,t)}}{\sqrt{2}}[e^{i\pi/4}e^{i\beta\pi}c_1(X-2\pi,t-1) + e^{-i\pi/4}c_2(X-\pi,t-1)]. \tag{A4}$$

One can show that the amplitudes are periodic functions, i.e., $c_{1,2}(X,t)=c_{1,2}(X+2\pi,t)$ by combining Eq. (33) with the equality $\psi_\beta(X-2\pi,t)=e^{-i2\pi\beta}\psi_\beta(X,t)$. This then leads to the matrix expression

$$\begin{pmatrix} c_1(X,t) \\ c_2(X+\pi,t) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi}e^{-i\pi/4} & e^{-i\phi}e^{i\pi/4}e^{-i\beta\pi} \\ e^{i\phi}e^{i\pi/4}e^{i\beta\pi} & e^{i\phi}e^{-i\pi/4} \end{pmatrix} \times \begin{pmatrix} c_1(X,t-1) \\ c_2(X+\pi,t-1) \end{pmatrix}. \tag{A5}$$

3. The case $\beta=0$

The explicit expression for the amplitudes $c_{1,2}$ for $\beta=0$ is obtained as follows. The diagonalization matrix $\mathbf{P}$ is formed with the eigenvectors of $e^{i\pi/4}\mathbf{M}_t$. For $\lambda=e^{\pm i\Theta}$ these eigenvectors are $(-ie^{-i\phi}, e^{-i\phi}-\sqrt{2}e^{\pm i\Theta})^T$. Thus

$$\mathbf{P} = \begin{bmatrix} -ie^{-i\phi} & -ie^{-i\phi} \\ e^{-i\phi}-\sqrt{2}e^{i\Theta} & e^{-i\phi}-\sqrt{2}e^{-i\Theta} \end{bmatrix}. \tag{A6}$$

The amplitudes c_1 and c_2 at time t, can be obtained by an explicitly development corresponding to Eq. (39), leading to Eq. (40).

APPENDIX B: CALCULUS OF THE AVERAGE MOMENTUM IN THE $\hbar=\pi$ CASE

Starting from Eq. (31) and changing $X+wt$ into X, we easily obtain a recursion relation for the derivative

$$\frac{\partial}{\partial X}\psi_\beta(X,t) = i\kappa \sin X e^{-i\kappa\cos X}\psi_\beta(X,t) + \frac{e^{-i\kappa\cos X}}{\sqrt{2}}\left(e^{-i\pi/4}\frac{\partial}{\partial X}\psi_\beta(X-w,t-1) + e^{i\pi/4}e^{i\beta\pi}\frac{\partial}{\partial X}\psi_\beta(X-w-\pi,t-1)\right). \tag{B1}$$

Using this expression for calculating the average momentum produces two terms p_1 and p_2. The first one is given by

$$p_1 = K\int_{-\pi}^{\pi} dX \sin X|\psi_\beta(X,t)|^2$$

and the term p_2 is

$$\begin{aligned} p_2 &= -\frac{i\hbar}{2}\int_{-\pi}^{\pi} dX[e^{i\pi/4}\psi_\beta^*(X-w,t-1) + e^{-i\pi/4}e^{-i\beta\pi}\psi_\beta^*(X-w-\pi,t-1)] \\ &\quad\times\left(e^{-i\pi/4}\frac{\partial}{\partial X}\psi_\beta(X-w,t-1) + e^{i\pi/4}e^{i\beta\pi}\frac{\partial}{\partial X}\psi_\beta \times(X-w-\pi,t-1)\right) \\ &= -\frac{i\hbar}{2}\int_{-\pi}^{\pi} dX\left(\psi_\beta^*(X,t-1)\frac{\partial}{\partial X}\psi_\beta(X,t-1) + \psi_\beta^*(X-\pi,t-1)\frac{\partial}{\partial X}\psi_\beta(X-\pi,t-1)\right) \\ &\quad+\frac{\hbar}{2}\int_{-\pi}^{\pi} dX\left(e^{i\beta\pi}\psi_\beta^*(X,t-1)\frac{\partial}{\partial X}\psi_\beta(X-\pi,t-1) - e^{-i\beta\pi}\psi_\beta^*(X-\pi,t-1)\frac{\partial}{\partial X}\psi_\beta(X,t-1)\right). \end{aligned}$$

In the last equality we replaced $(X-w)$ by X; as the integration is over one period, the integration limits can be kept. One recognizes $\langle P\rangle_\beta(t-1)$ in the first term, while the second term cancels out. The recursion relation for the momentum is thus

$$\langle P\rangle_\beta(t) = \langle P\rangle_\beta(t-1) + K\int_{-\pi}^{\pi} dX \sin X|\psi_\beta(X,t)|^2. \tag{B2}$$

We now use the decomposition Eq. (33) to transform Eq. (B2) into

$$\langle P\rangle_\beta(t) = \langle P\rangle_\beta(t-1) + iK\cos(\beta\pi)\int_{-\pi}^{\pi} dX \sin(X+wt) \times[c_1^*(X,t)\psi_\beta^*(X,0) + c_2^*(X,t)\psi_\beta^*(X-\pi,0)] \times[c_1(X,t)\psi_\beta(X,0) + c_2(X,t)\psi_\beta(X-\pi,0)].$$

If the initial wave function has a narrow distribution centered around position X_0, we can neglect overlaps of functions $\psi_\beta(X,0)$ and $\psi_\beta(X+\pi)$ [54], producing

$$\langle P\rangle_\beta(t) = \langle P\rangle_\beta(t-1) + K\int_{-\pi}^{\pi} dX\,\sin(X+wt)[|c_1(X,t)|^2 - |c_2(X-\pi,t)|^2]|\psi_\beta(X,0)|^2. \tag{B3}$$

Finally, using the fact that $\psi_\beta(X,0)$ is much narrower then any other factor, and assuming $\int_{-\pi}^{\pi}dX|\psi_\beta(X,0)|^2=1$ [55] gives

$$\langle P\rangle_\beta(t) = \langle P\rangle_\beta(t-1) + K\,\sin(X_0+wt)[1-2|c_2(X_0-\pi,t)|^2]. \tag{B4}$$

APPENDIX C: MOMENTUM AVERAGED OVER THE QUASIMOMENTUM

The momentum averaged over the quasimomentum

$$\langle P\rangle = \hbar\!\!\!/\sum_n \int_{-1/2}^{1/2} d\beta(n+\beta)|\tilde{\psi}_\beta(n)|^2 \tag{C1}$$

can be related to quantum average $\langle P\rangle_\beta$ in the following way: From the definition of Sec. II one has

$$\tilde{\psi}_\beta(n) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\psi_\beta(X)e^{-i(\beta+n)X}dX. \tag{C2}$$

Therefore,

$$\begin{aligned}(n+\beta)\tilde{\psi}_\beta(n) &= \frac{i}{\sqrt{2\pi}}\int_{-\pi}^{\pi}dX\psi_\beta(X)\frac{\partial}{\partial X}e^{-i(\beta+n)X}\\ &= \frac{i}{\sqrt{2\pi}}[\psi_\beta(X)e^{-i(\beta+n)X}]_{-\pi}^{\pi}\\ &\quad - \frac{i}{\sqrt{2\pi}}\int_{-\pi}^{\pi}dXe^{-i(\beta+n)X}\frac{\partial}{\partial X}\psi_\beta(X).\end{aligned} \tag{C3}$$

The first term on the right-hand side vanishes [$\psi_\beta(X)e^{-i\beta X}$ is 2π periodic], thus

$$\langle P\rangle = -\frac{i\hbar\!\!\!/}{\sqrt{2\pi}}\sum_n\int d\beta\tilde{\psi}^*_\beta(n)\int_{-\pi}^{\pi}dXe^{-i(\beta+n)X}\frac{\partial}{\partial X}\psi_\beta(X).$$

Using $\sum_n e^{-in(X-X')}=\sum_n\delta(X-X'-2n\pi)$ one obtains

$$\langle P\rangle = -i\hbar\!\!\!/\int d\beta\int_{-\pi}^{\pi}dX\psi^*_\beta(X)\frac{\partial}{\partial X}\psi_\beta(X) = \int_{-1/2}^{1/2}d\beta\langle P\rangle_\beta, \tag{C4}$$

where

$$\langle P\rangle_\beta = \int_{-\pi}^{\pi}dX\psi^*_\beta(X)[P\psi_\beta(X)]. \tag{C5}$$

For the second momentum of P, an analogous development leads to

$$\langle P^2\rangle = \sum_n\int d\beta(n+\beta)^2|\tilde{\psi}_\beta(n)|^2 = \int d\beta\langle P^2\rangle_\beta, \tag{C6}$$

where

$$\langle P^2\rangle_\beta = \int_{-\pi}^{\pi}dX[P\psi_\beta(X)]^*[P\psi_\beta(X)] \tag{C7}$$

(and, in general, $\langle P^k\rangle=\int d\beta\langle P^k\rangle_\beta$ for any integer k).

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